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# Cycle Covering of Binary Matroids

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Motivated by some problems which had been left open in a previous paper [M. Tarsi, *J. Combin. Theory Ser. B* **39** (1985), 346–352], we present the following results:

1. Every bridgeless binary matroid with no $F_7^*$ minor (in particular every regular matroid) had a cycle in which every element is covered exactly 4 times.

2. The cycle double cover conjecture for graphs is equivalent to a simular conjecture for binary matroids with no $F_7^*$ minor.

3. We give the lowest upper bounds for the length of the shortest cycle cover of cographic matroids which admit a $k$-nowhere zero flow.

4. We generalize the concept of nowhere zero flows to binary, non-regular matroids and relate it to the length of a shortest cycle cover. © 1989 Academic Press, Inc.

## 1. INTRODUCTION

In this paper we deal with the problems stated in Section V of [8]. We shall use the notation of [8] and refer the reader to Sections I, II, and III of that paper for the definitions of matroid orientation, nowhere zero flow (NZF), cycle cover, the lenght of a cycle cover, etc.

In particular we use the same symbol to refer to both a matroid and its underlying element set. The main subject of [8] is the cycle cover ratio $r(M)$ of a binary matroid $M$, defined as the lenght of the shortest cycle cover of $M$ divided by $|M|$. (We denote by $r(S)$ the lenght of a specific cycle cover $S$ of $M$ divided by $|M|$.) In [8] the parameter

$$s(k) = \sup\{r(M) \mid M \text{ is a regular matroid which admits a } k\text{-NZF}\}$$

is defined and studied. The following problems are stated in the last section of [8] and are left open:

Problem 1. What is the exact value of $s(5)$?



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Problem 2. Is there a constant $c$ such that $s(k) \leqslant c$ for every $k$. In particular, is 2 such a constant?

Problem 3. What is the value of a parameter, similar to $s(k)$, where the domain is restricted to contain only cographic matroids? In other words, what is the least upper bound for the cycle cover ratio of cographic matroids with a $k$-NZF (obtained by covering the edge set of a $k$-colorable graph by cocycles)?

Problem 1 has recently been solved by Jamshy, Raspaud, and Tarsi [5], who proved $s(5) = \frac{8}{5}$.

Problem 2 is discussed in section 2 of this paper, where $s(k) \leqslant 4$ is proved for every $k \geqslant 2$. It is also shown that 4 can be replaced by 2 if the cycle double cover conjecture (see [3]) is true. This is done by extending the conjecture, originally stated for graphs, to all binary matroids which have no $F_7^*$ (the dual of the Fano matroid) minor. The main tool used in Section 2 is Seymour's decomposition theorem for regular matroids [6].

Problem 3 is solved in Section 3. It was independently answered by Jaeger, Khelladi, and Mollard [4], however, we give a very much shorter proof for the same result: The least upper bound for the shortest cut (cocycle) cover ratio of a $k$-colorable graph is $(2-2/k)$ for $k \notin \{4, 8\}$; it is $\frac{4}{3}$ and $\frac{12}{7}$ for $k=4$ and $k=8$, respectively. The result in [4] is more complete than ours since it contains an explicit description of all the shortest cut covers of the complete graph $K_k$, while in our proof this is done only for $k \geqslant 9$ where there is only one such cover (up to the selection of a specific vertex).

Motivated by problem 2 we give, in section 4, a natural generalization of a $2^n$–NZF in a binary, not necessarily regular, matroid. We define $s_b(2^n)$ for binary matroids in the same way that $s(k)$ is defined for regular matroids. It turns out that while $s(k)$ is always at most 4, the generalized parameter $s_b$ is not bounded above and the same recursion used in [8] to bound $s(k)$ above provides the exact value of $s_b\,(2^n)$.

## 2. Covering Regular Matroids by Cycles

In [8] it is conjectured that $s(k)$ is bounded above and the constant 2 is suggested as a possible upper bound. (If that is true then, as we show in section 3, 2 is the least upper bound.) Let us define an $l$-cycle cover of a binary matroid $M$ to be a family of cycles in which every element of $M$ appears exactly $l$ times. If $M$ is graphic then it has a 4-cycle cover (see [2]). A cographic matroid obviously has a 2-cycle cover, consisting of the vertex cocycles of its associated graph. P. D. Seymour in his now classical paper [6] proved that every regular matroid is either graphic and bridgeless (by bridgeless we mean contains no coloop), or bridgeless cographic, or isomorphic to a specific matroid $R_{10}$, or it is the 1-, 2- or 3-sum (see [6] for the definitions) of smaller regular matroids.

We now use this theorem to prove that the smallest bridgeless regular matroid $M$ for which there is no $2k$-cycle cover is graphic.

Let $M$ be a bridgeless non-graphic regular matroid such that every smaller (with respect to the number of elements) bridgeless regular matroid has a $2k$-cycle cover. We apply the decomposition theorem to the dual matroid $M^*$. If $M^*$ is either graphic (that is, $M$ is cographic) or isomorphic to $R_{10}$ then $M$ has a 2-cycle cover and thus a $2k$-cycle cover. If not then according to Seymour's theorem $M^*$ is respresentable as a 1-, 2-, or 3-sum of two of its proper minors. The duals of these minors are bridgeless and regular and thus each has a $2k$-cycle cover by the minimality of $M$. It is a simple routine to show that if two binary matroids $M_1$ and $M_2$ each have a $2k$-cycle cover then their 1- and 2-sums also have $2k$-cycle covers (1- and 2-sum are self dual operations so we get the required results for the dual of their sum). Regarding the case of a 3-sum: If $M^*$ is the 3-sum of $M_1^*$ and $M_2^*$, let $\{e, f, g\}$ be their common circuit. Let $\{C_1, ..., C_m\}$ be a $2k$-cycle cover of $M_1$. Since $\{e, f, g\}$ is a cocircuit of $M_1$, precisely $k$ of $C_1, ..., C_m$ contain $e, f$ (and not $g$), and similarly for $f$, $g$ and for $g$, $e$. Let $C_1, ..., C_k$ contain $e, f$, $C_{k+1}, ..., C_{2k}$ contain $f$, $g$ and $C_{2k+1}, ..., C_{3k}$ contain $g$, $e$. Let $\{D_1, ..., D_n\}$ be a $2k$-cycle cover of $M_2$, numbered similarly. Then for $1 \leqslant i \leqslant 3k$, $C_i + D_i$ is a disjoint union of circuits of $M$ and these together with $C_{3k+1}, ..., C_m, D_{3k+1}, ..., D_n$ provide a $2k$-cycle cover of $M$, as required.

Another theorem proved by Seymour in [6] deals with binary matroids with no $F_7^*$ (the dual of the Fano matroid $F_7$) minor: such a matroid is either regular or isomorphic to $F_7$, or it is the 1- or 2-sum of two of its proper minors (see also [7, (6.1)] where the result is stated in a dual form). Since $F_7$ is Eulerian a direct consequence of the last discussion is:

LEMMA 2.1. *A smallest bridgeless binary matroid with no $F_7^*$ minor for which there is no $2k$-cycle cover is graphic.*

Lemma 2.1 for $k = 2$ together with [2, Proposition 6] yields:

THEOREM 2.1. *Every bridgeless binary matroid with no $F_7^*$ minor has a 4-cycle cover, in particular $s(k) \leqslant 4$ for every $k$.*

Although the cycle cover ratio for graphic, as well as $R_{10}$ and cographic, matroids is bounded by 2, we do not see how to prove the same upper bound for general regular matroids. However, that is very easy if the cycle double cover conjecture is true. Lemma 2.1 for $k = 1$ yields the equivalence of the cycle double cover conjecture for graphs and the following, apparently stronger statement:

*Conjecture* (equivalent to the cycle double cover conjecture for graphs). Every binary matroid with no $F_7^*$ minor (in particular every regular matroid) has a 2-cycle cover.

## 3. Shortest Cut Cover for Graphs

As we have just seen the validity of the cycle double cover conjecture would imply $s(k) \leqslant 2$ for every $k$. In this section we prove that 2 cannot be replaced by any smaller number since for every $\varepsilon > 0$ we have a regular matroid $M$ with $r(M) > 2 - \varepsilon$.

Let us start with some more notation:

A *shortest cut cover* of a graph $G$ is defined as a shortest cycle cover of the cographic matroid of $G$. We use the term *cut* to refer to a cocycle of a graph. We shall denote by $c(A)$, $A \subset V$, the cut in $G = (V, E)$ consisting of all the edges $(x, y)$ such that $x \in A$, $y \in V - A$. A cut of the form $c(\{x\})$ will be called *a star*.

A *standard cut cover* of the complete graph $K_n$ is a cut cover which consists of $n-1$ stars. Hence, the lenght of a standard cut cover of $K_n$ is $(n-1)^2$.

The *weight* of an edge $e$ with respect to a cut-cover $S$ is the number of cuts in $S$ which contain $e$.

The *weight* of a vertex $v$ with respect to a cut-cover $S$ is the sum of the weights of the edges which are incident to $v$. We shall denote it by $w_S(v)$.

By "deletion of a vertex $u$ from a cut-cover $S$" we mean the deletion of all the edges incident to $u$ from every cut in $S$.

The following result has also been obtained independently by N. Alon (private communication) and by Jaeger, Khelladi, and Mollard [4].

Lemma 3.1. *Any shortest cut cover of $K_n$ for $n > 8$ is standard.*

The lemma can be checked for $n = 9$ by a routine, although quite long, case analysis. We proceed by induction on $n$. Let us assume the lemma is true for some $n \geqslant 9$ but that there exists a non standard shortest cut cover $S$ for $K_{n+1}$. By $S'$ we denote the cut cover of $K_n$, obtained by the deletion of a vertex $v$ from $S$. Clearly

$$l(S') = l(S) - w_S(v).$$

A nonstandard cover $T$ for $K_{n+1}$ which by deleting a vertex $a_{n+1}$ becomes a standard cover $T'$ for $K_n$ is of the form

$$T = \{c(\{a_{n+1}, a_1\}), c(\{a_{n+1}, a_2\}), ..., c(\{a_{n+1}, a_k\}),\\ c(\{a_{k+1}\}), ..., c(\{a_{n-1}\})\}$$

($k \geqslant 2$ otherwise the edge $(a_1, a_{n+1})$ is not covered) and by deleting $a_{n+1}$:

$$T' = \{c(\{a_1\}), c(\{a_2\}), ..., c(\{a_{n-1}\})\}.$$

It can be easily checked that $l(T) = n^2 + nk - n - 2k$ $(k > 1, n > 8)$ is greater than $n^2$, the lenght of a standard cover, and hence $T$ is not shortest. Thus, if $S$ is shortest and nonstandard, then $S'$ is nonstandard.

Suppose

$$l(S) = n^2 - d \qquad (d \geqslant 0).$$

The sum of the weights of all the vertices is twice the lenght of $S$, $2(n^2 - d)$.

The average weight per vertex is

$$2\frac{n^2 - d}{n+1} = 2n - 2\frac{n+d}{n+1} > 2n - d - 2.$$

Hence there exists at least one vertex $u$ for which $w_S(u) \geqslant 2n - d - 1$. Deleting such a vertex $u$ from $S$ we obtain a cut cover $S'$ with

$$l(S') = l(S) - w_S(u) \leqslant n^2 - d - (2n - d - 1) = (n-1)^2.$$

Thus $S'$ is a shortest cut cover for $K_n$ and is nonstandard, contradicting the induction hypothesis.

For $3 < n < 9$ there exist shortest nonstandard cut covers. In fact for $n = 4$ and $n = 8$, the minimal cut cover is one edge shorter than the standard (see [4, 8]). For $n = 5, 6, 7$ there are nonstandard shortest cut covers of lenght $(n-1)^2$.

A consequence of this lemma is:

LEMMA 3.2. *The cographic matroids of $K_n$ for $n \notin \{4, 8\}$ have a shortest cover $S$ such that*:

$$r(S) = 2 - \frac{2}{n}.$$

Define

$$s_{cg}(k) = \sup\{r(M) \mid M \text{ is cographic and admits a } k-\text{NZF}\}.$$

We can prove now:

THEOREM 3.1.

$$s_{cg}(4)=\frac{4}{3}, \qquad s_{cg}(8)=\frac{12}{7}, \qquad \textit{and} \quad s_{cg}(k)=2-\frac{2}{k} \qquad \textit{for} \quad k \notin \{4, 8\}.$$

*Proof.* Let $G$ be a $k$-colorable graph with a good coloring of its vertices with the colors $1, 2, \ldots k$. For each color $i$ define the cut $c_i$ to be the set of all edges for which exactly one end-vertex is colored $i$. The $k-1$ $c_i$'s of smallest cardinality form a cut cover $S$ of $G$ with $r(S) \leqslant 2-2/k$. Combined with Lemma 3.2 the proof is complete for $k \notin \{4, 8\}$. The cases $k=4$ and $k=8$ are settled by $s(4)=\frac{4}{3}$, $s(8)=\frac{12}{7}$ (see [8]), the obvious fact that $s_{cg}(k) \leqslant s(k)$, and the observations $r(M^*(K_4))=\frac{4}{3}$, $r(M^*(K_8))=\frac{12}{7}$. ∎

## 4. FLOW AND SHORTEST CYCLE COVER IN BINARY MATROIDS

The definition of a $G$-flow in regular matroids can be generalized to binary matroids as follows: Let $M$ be a binary matroid and $G$ an Abelian group. Define A $G$-flow in $M$ to be a function, $f$: $M \rightarrow G$, satisfying

$$\sum_{x \in C^*} f(x)=0$$

over any cocycle $C^*$ of $M$ (additive notation is used for $G$ and $0 \in G$ is its zero element). The *support* of A $G$-flow $f$ is

$$\|f\|=\{x \in M \mid f(x) \neq 0\}.$$

A *G-nowhere-zero flow*, $G$-NZF, is a $G$-flow $f$ for which $\|f\|=M$. These definitions are similar to those given for flows in an undirected network and differ from the definitions we use for regular oriented matroids. However, in case where $M$ is regular and $G=(Z_2)^n$, these definitions are equivalent to those given in [8] and that is the only case we deal with in this section.

Following [8] we define for $k=2^n$, $s_b(k)$ to be the smallest $s$ such that every binary matroid with a $(Z_2)^n-$NZF has a cycle cover $S$ with $r(S) \leqslant s$. The proof of Theorem I of [8] is easily adapted to this case yielding for $k$ and $t$ powers of 2

$$s_b(kt) \leqslant \frac{s_b(k)(kt-t)+s_b(t)(kt-k)}{kt-1} \tag{4.1}$$

which leads by induction to

$$s_b(2^n) \leqslant \frac{n\,2^{n-1}}{2^n - 1}. \tag{4.2}$$

For a detailed proof see [8]. In [8] regularity is used for the equivalence of $Z_{kt} - \mathrm{NZF}$ and $Z_k \times Z_t - \mathrm{NZF}$ which is trivial in the case we are dealing with here. Notice that the existence of a $Z_2^n - \mathrm{NZF}$ is equivalent to the existence of a cycle cover consisting of $n$ cycles. This also provides an upper bound to justify the existence of the least upper bound $s_b(k)$.

Equation (4.2) can also be proved directly starting with a cover $S$ of a binary matroid by $n$ cycles and taking the average lenght of all regular linear transformations of $S$ over $GF_2$; see [1, Proposition 3.1].

Let $(GF_2)^n$ stand for the $n$-dimensional vector space over the field $GF_2$. Denote the linear independence matroid of $(GF_2)^n$ with the zero vector excluded by $F_{2^n-1}$ (for $n=3$ it is the Fano matroid $F_7$) and its dual by $F^*_{2^n-1}$. The circuits of $F^*_{2^n-1}$ are the complements of the hyperplanes of $(GF_2)^n$. These hyperplanes are the $n-1$-dimensional subspaces; hence each circuit of $F^*_{2^n-1}$ contains $2^{n-1}$ elements. There are $2^n - 1$ hyperplanes in $(GF_2)^n$ (each is defined by its orthogonal space which contains a single non-zero vector). $2^n - 1$ is also the number of cycles in the cycle space of $F^*_{2^n-1}$, hence every cycle of that matroid is a circuit. It follows that a cycle cover of $F^*_{2^n-1}$ corresponds (taking the complements) to a collection of hyperplanes of $(GF_2)^n$ whose intersection is a single point—the zero vector. It is a well-known algebraic fact that there exists such a collection consisting of $n$ hyperplanes and that there is no one with less then $n$. Thus, a shortest cycle cover of $F^*_{2^n-1}$ consists of $n$ cycles each of $2^{n-1}$ elements. Hence $r(F^*_{2^n-1}) = n\,2^{n-1}/2^n - 1$. The last equality together with 4.2 yields:

$$s_b(2^n) = \frac{n\,2^{n-1}}{2^n - 1}.$$

To conclude: Although $s(k) \leqslant 4$ for every $k$, Theorem 1 of [8] gives, in some sense, the best possible result in the case where binary matroids are considered.

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